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The mathematics of generalized Fibonacci sequences: Binet's formula and identities

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Abstract. This article considers a generalized Fibonacci sequence $\{V_n\}$ with general initial conditions, $V_0 = a$, $V_1 = b$, and a versatile recurrence relation $V_n = pV_{n-1} + qV_{n-2}$, where $n \geq 2$ and a, b, p and q are any non-zero real numbers. The generating function and Binet formula for this generalized sequence are derived. This generalization encompasses various well-known sequences, including their generating functions and Binet formulas as special cases. Numerous new properties of these sequences are studied and investigated.

1. Introduction

Several generalizations of the Classical Fibonacci sequence are available in the literature. Some authors ([2, 11, 13–15, 19, 25]) have altered the initial conditions, while others ([4, 16–18, 20–22, 24, 26, 27]) have generalized the recurrence relation. Author [8] presented an analytic proof of the classical Fibonacci recursive formula. Fibonacci and Lucas Numbers are studied in [9] and [10] using different approaches. Some generalized formulas of Fibonacci are also obtained and studied in [7].

This article considers a generalization of the Fibonacci sequence by modifying both the initial conditions and the recurrence relation. By applying appropriate restrictions to the parameters, this generalization encompasses the Classical Fibonacci, Lucas, Pell, Pell-Lucas, modified Pell, Goksal Bilgici, Jacobstal, and Jacobstal-Lucas sequences. These generalizations exhibit numerous fascinating properties and have applications in various fields of mathematical sciences, including quasi-crystals, algebra, graph theory, and computer algorithms ([1, 3, 5, 6, 14, 21, 23]).

Definition 1 (Fibonacci's Generalization)**.** We define the generalization of the Fibonacci sequence $\{V_n\}_{n=0}^{\infty}$ by the following recurrence relations:

$$V_n = pV_{n-1} + qV_{n-2} \tag{1}$$

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with the initial conditions,$V_0 = a, V_1 = b$, a, b, p and q are any non–zero real numbers.

The expression for $\{V_n\}$ in (1) is true [15] for every integer $n \geq 2$.

Definition 2 (Fibonacci's Recursion Generalization representation)**.** Equivalently, expression for $\{V_n\}$ in (1) can be expressed as:

$$
\begin{aligned}
V_n &= \left(\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} p^{n-2i-1} q^i \right) b \\
&\quad + \left(\sum_{i=0}^{\left[\frac{n-2}{2}\right]} \binom{n-i-2}{i} p^{n-2i-2} q^i \right) a, \quad n > 0.,
\end{aligned} \tag{2}
$$

a, b, p and q are any non–zero real numbers.

Remark 1. If $n \geq 2$ is any integer, and $p = 1$, $q = 1$ and $V_0 = F_0 = 0$, $V_1 = F_1 = 1$ in (2), then

$$
V_n = \left(\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} p^{n-2i-1} q^i \right), \quad n > 0.
$$

Evidently, for $(p, q) = (1, 1)$ and $(a, b) = (1, 1)$, $(p, q) = (1, 1)$ and $(a, b) = (2, 1)$, $(p, q) = (2, 1)$ and $(a, b) = (2, 1)$, $(p, q) = (2, 1)$ and $(a, b) = (2, 2)$, $(p, q) = (1, 2)$ and $(a, b) = (0, 1)$, $(p, q) = (1, 2)$ and $(a, b) = (2, 1)$ and $(p, q) = \left(2s, t^2 - s\right)$ and $(a, b) = (0, 1)$, $(p, q) = \left(2s, t^2 - s\right)$ and $(a, b) = (2, 2s)$, where s and t are any non – zero real numbers, the sequence $\{V_n\}$ defined in (1) and (2) becomes the Classical Fibonacci, Lucas, Pell, Modified Pell, Pell-Lucas Jacobsthal, Jacobsthal-Lucas and Goksal Bilgici sequences, respectively.

The well-known Classical **Fibonacci sequence** is defined by the recurrence relation and initial conditions is

Definition 3 (Classical Fibonacci's Recursion)**.** $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$.

Some of the well–known generalizations of the classical Fibonacci sequences are:

- The **Lucas sequence** is defined as: $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, $L_0 = 2$, $L_1 = 1$.
- The **Pell sequence** is defined as: $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$, $P_0 = 2$, $P_1 = 1$. $q_n = 2q_{n-1} + q_{n-2}$, $n \geq 2$, $q_0 = 1$, $1_1 = 1$.
- The **Pell-Lucas sequence** is defined is defined as: $Q_n = 2Q_{n-1} + Q_{n-2}$, $n \geq 2$, $Q_0 = 2$, $Q_1 = 2$.

- The **Goksal Bilgici sequences** are defined by the recurrence relation, $f_n = 2aq_{n-1} + (b^2 - a)q_{n-2}$, $n \geq 2$, $q_0 = q$, $q_1 = 1$.
  and $l_n = 2al_{n-1} + (b^2 - a)l_{n-2}$, $n \geq 2$, $l_0 = 2$, $q_1 = 2a$.
- The **Jacobsthal sequences** are defined as: $J_n = J_{n-1} + 2J_{n-2}$, $n \geq 2$, $J_0 = 0$, $J_1 = 1$.
- The **Jacobsthal-Lucas sequences** are defined as: $L_n = L_{n-1} + 2J_{n-2}$, $n \geq 2$, $L_0 = 2$, $L_1 = 1$.

Theorem 1. *If $n \geq 2$ is any integer and $\{V_n\}$ defined in (2), then*

$$\sum_{i=0}^{n} \binom{n}{r} V_r = \frac{1}{(\alpha-\beta)p^n} \left(p - q + \alpha^2\right)^n (V_1 - \beta V_0) - \left(p - q + \beta^2\right)^n (V_1 - \alpha V_0).$$

Proof. Using the definition of $\{V_n\}$ in (2), we have

$$\sum_{i=0}^{n} \binom{n}{r} V_r = \frac{1}{(\alpha-\beta)} \left[(1+\alpha)^n (V_1 - \beta V_0) - (1+\beta)^n (V_1 - \alpha V_0)\right].$$

Now using

$$1 + \alpha = \left(p - q + \alpha^2\right)/p, \qquad 1 + \beta = \left(p - q + \beta^2\right)/p,$$

we have

$$\sum_{i=0}^{n} \binom{n}{r} V_r = \frac{1}{(\alpha-\beta)p^n} \left(p - q + \alpha^2\right)^n (V_1 - \beta V_0) - \left(p - q + \beta^2\right)^n (V_1 - \alpha V_0). \qquad \square$$

Corollary 1. *If $p = 1, q = 1$ and $V_0 = F_0 = 0$, $V_1 = F_1 = 1$ is substituted in the above expression, then*

$$\sum_{i=0}^{n} \binom{n}{r} F_r = \frac{1}{(\alpha-\beta)} \left(\alpha^2\right)^n - \left(\beta^2\right)^n = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = F_{2n},$$

which is in agreement with (Lucas) (page 157, Theorem 12.5 of Thomas Koshy [15]).

Theorem 2. *Ifn ≥ 2, is any integer and $\{V_n\}$ defined in (2), then*

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r V_r = \frac{1}{(\alpha-\beta)} \left[(1-\alpha)^n (V_1 - \beta V_0) - (1-\beta)^n (V_1 - \alpha V_0)\right].$$

Proof. Using the definition of $\{V_n\}$ from (2), we have

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r V_r = \frac{1}{(\alpha-\beta)} (1-\alpha)^n (V_1 - \beta V_0) - \frac{1}{(\alpha-\beta)} (1-\beta)^n (V_1 - \alpha V_0)$$

$$= \frac{1}{(\alpha-\beta)} \left[(1-\alpha)^n (V_1 - \beta V_0) - (1-\beta)^n (V_1 - \alpha V_0)\right],$$

where α and β are the roots of the equation $x^2 - px - q = 0$. $\square$

Corollary 2. *If $p = 1, q = 1$ and $V_0 = F_0 = 0$, $V_1 = F_1 = 1$ are substituted in the above expression, then*

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r F_r = \frac{1}{(\alpha - \beta)} \left[(-\beta)^n - (-\alpha)^n\right]$$

$$= (-1)^{n-1} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) = (-1)^{n-1} F_n,$$

where α and β are the roots of the equation $x^2 - x - 1 = 0$.

Theorem 3 (Generalized Generating Functions). *The generalized generating function of the sequence defined in* (1) *is*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{a + (b - pa)x}{(1 - px - qx^2)}.$$

Proof. Let $V(x)$ be the generating function of (1),

$$V(x) = \sum_{n=0}^{\infty} V_n x^n, \tag{3}$$

then

$$V(x) - pxV(x) - qx^2V(x) = \sum_{n=0}^{\infty} V_n x^n - px \sum_{n=0}^{\infty} V_n x^n - qx^2 \sum_{n=0}^{\infty} V_n x^n, \tag{4}$$

On simplification, we have

$$\left(1 - px - qx^2\right) V(x) = V_0 + (V_1 - pV_0)\, x$$

Hence

$$\sum_{n=0}^{\infty} V_n x^n = \frac{a + (b - pa)x}{(1 - px - qx^2)}. \tag{5}$$

□

Theorem 4 (Generalized Binet's formula). *The generalized Binet's formula for the sequence $\{V_n\}$ in* (2) *is*

$$V_n = a \sum_{k=0}^{n} \alpha^{n-k} \beta^k + (b - a(\alpha + \beta)) \sum_{k=0}^{n} \alpha^{n-k-1} \beta^k$$

where $\alpha, \beta = \frac{p \pm \sqrt{p^2 + 4q}}{2}$ are the roots of the equation $x^2 - px - q = 0$.

Proof. Consider partial fraction decomposition of the right-hand side of the generating function of the sequence defined in (5)

$$\frac{a + (b - pa)\, x}{(1 - px - qx^2)} \equiv \frac{(\alpha a + b - pa)}{(\alpha - \beta)(1 - \alpha x)} - \frac{(\beta a + b - pa)}{(\alpha - \beta)(1 - \beta x)}.$$

On simplification we have

$$= \frac{a}{(\alpha - \beta)} \left[\frac{\alpha}{(1 - \alpha x)} - \frac{\beta}{(1 - \beta x)}\right] + \frac{(b - pa)}{(\alpha - \beta)} \left[\frac{1}{(1 - \alpha x)} - \frac{1}{(1 - \beta x)}\right]$$

we have

$$\sum_{n=0}^{\infty} V_n x^n = \left[a\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+(b-pa)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\right]x^n. \tag{6}$$

Thus, the generalized form of the Binet formula for the generalized Fibonacci sequence $\{V_n\}_{n=0}^{\infty}$ is

$$V_n = a\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+(b-pa)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right), \tag{7}$$

or equivalently expression (3) can be expressed as

$$V_n = a\sum_{k=0}^{n}\alpha^{n-k}\beta^k + (b-a(\alpha+\beta))\sum_{k=0}^{n}\alpha^{n-k-1}\beta^k, \tag{8}$$

where α and β are the roots of the equation $x^2-px-q=0$. □

2. Special Cases of generating functions and Binet formula

Case 1: Fibonacci sequences

Substituting $p=1, q=1$ and $a=F_0=0$, $b=F_1=1$ in (3) and (4), subsequently, the generating function and Binet formula simplifies to:

$$F_n = F_{n-1}+F_{n-2}\,(V_n = pV_{n-1}+pV_{n-2}),\ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} F_n x^n = \frac{F_0+(F_1-pF_0)\,x}{(1-px-qx^2)} = \frac{1}{(1-x-x^2)}, \tag{9}$$

$$V_n = 0\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+(1-0)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \tag{10}$$

Therefore, (9) and (10) represent the generating function and Binet formula for the well-known classical Fibonacci sequences.

Case 2: Lucas sequence

Substituting $p=1, q=1$ and $a=l_0=2$, $b=l_1=1$ in (3), then the generating function in (3), simplifies to:

$$l_n = 2l_{n-1}+l_{n-2},\ n\geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2-x}{(1-x-x^2)}, \tag{11}$$

$$l_n = \alpha^n+\beta^n, \tag{12}$$

where $\alpha,\beta=\frac{1\pm\sqrt{5}}{2}$. Thus, (11) and (12) represent the generating function and Binet formula for the well-known Lucas sequence. This is in agreement with the generating function for Lucas sequence.

Case 3: Pell sequence

Substituting $p=2, q=1$ and $a=P_0=0$, $b=P_1=1$ in (3), then the generating function in (3), simplifies to:

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1}{(1-2x-x^2)}. \tag{13}$$

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha, \beta = 1 \pm \sqrt{2}. \tag{14}$$

Thus, (13) and (14) represent the generating function and Binet formula for the Pell sequence. This is in agreement with the generating function and Binet formula for Pell sequence.

Case 4:Modified Pell sequence

Substituting $p = 2, q = 1$ and $a = f_0 = 1$, $b = f_1 = 1$ in (3), then the generating function in (3), simplifies to:

$$f_n = 2f_{n-1} + f_{n-2},\ n \geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} f_n x^n = \frac{1-x}{(1-2x-x^2)}. \tag{15}$$

$$q_n = \alpha^n + \beta^n, \quad \alpha, \beta = 1 \pm \sqrt{2}. \tag{16}$$

Thus, (15) and (16) represent the generating function and Binet formula for the Modified Pell sequence. This is in agreement with the generating function and Binet formula for Modified Pell sequence.

Case 5: Pell-Lucas sequence Substituting $p = 2, q = 1$ and $a = Q_0 = 2$, $b = Q_1 = 2$ in (3), then the generating function in (3) simplifies to:

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2 \text{ is}$$

$$\sum_{n=0}^{\infty} f_n x^n = \frac{2-2x}{(1-2x-x^2)}. \tag{17}$$

$$\Rightarrow Q_n = 2\left[\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right] = 2q_n, \quad \alpha, \beta = 1 \pm \sqrt{2}. \tag{18}$$

Thus, (17) and (18) represent the generating function and Binet formula for the Pell-Lucas sequence. This is in agreement with the generating function and Binet formula for the Pell-Lucas sequence.

Case 6: Goksal Bilgici sequences

Substituting $p = 2a$, $q = b - a^2$ and $V_0 = f_0 = 0$, $V_1 = f_1 = 1$ and $p = 2a$, $q = b - a^2$ and $V_0 = l_0 = 2$, $V_1 = l_1 = 2a$ in (3), then the corresponding generating functions for the Goksal Bilgici sequences are:

$$f_n = 2af_{n-1} + (b - a^2)f_{n-2}, \quad n \geq 2$$

and

$$l_n = 2al_{n-1} + (b - a^2)l_{n-2}, \quad n \geq 2$$

are

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{(1-2ax-(b-a^2)x^2)} \tag{19}$$

and

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2-2ax}{(1-2ax-(b-a^2)x^2)}. \tag{20}$$

Thus, (19) and (20) are in agreement with the generating function for the well-known Goksal Bilgici sequences

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{21}$$

and

$$l_n = \alpha^n + \beta^n. \tag{22}$$

Thus, (21) and (22) are in agreement with the Binet's formul for the Goksal Bilgici sequences.
Case 7: Jacobsthal Sequences Substituting $p = 1, q = 2$ and $a = J_0 = 0$, $b = J_1 = 1$ in (3), then the corresponding generating functions for the Jacobsthal sequence is

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{(1-x-2x^2)}, \tag{23}$$

$$\begin{aligned} L_n &= 2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) - \frac{\alpha^n-\beta^n}{\alpha-\beta}, \\ L_n &= \alpha^n + \beta^n \left(\because 2\alpha - 1 = 3,\ 2\beta - 1 = -3,\ \alpha - \beta = 3\right). \end{aligned} \tag{24}$$

Here α and β are roots of the equation $x^2 - x - 2 = 0$, this formula is the same as is in [12]. Thus, (23) and (24) represent the generating function and Binet formula for the Jacobsthal Sequence. This is in agreement with the generating function and Binet formula for the Jacobsthal Sequence.
Case 8: Jacobsthal-Lucas Sequences Substituting $p = 1$, $q = 2$ and $a = J_0 = 2$, $b = J_1 = 1$ in (3), then the corresponding generating functions for the Jacobsthal-Lucas sequence are

$$\sum_{n=0}^{\infty} J_n x^n = \frac{2-x}{(1-x-2x^2)}, \tag{25}$$

$$L_n = \alpha^n + \beta^n \left(\because 2\alpha - 1 = 3,\ \ 2\beta - 1 = -3,\ \ \alpha - \beta = 3\right). \tag{26}$$

This is in agreement with the generating function and Binet formula for the Jacobsthal Sequence. Here α and β are roots of the equation $x^2 - x - 2 = 0$, this formula is the same as is in [12]. Thus, (25) and (26) represent the generating function and Binet formula for the Jacobsthal Sequence. □

Theorem 5. *If $n \geq 2$ is any integer and*

$$V_n = \left(\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} p^{n-2i-1} q^i \right) b + \left(\sum_{i=0}^{\left[\frac{n-2}{2}\right]} \binom{n-i-2}{i} p^{n-2i-2} q^i \right) a,$$

where a, b, p and q are any non–zero real numbers and

$$K_n = \begin{pmatrix} V_{n+2} & V_{n+1} \\ V_{n+1} & V_n \end{pmatrix},$$

then

$$\det K = (-1)^n \left(q^{n-2} b(b - ap) + q^{n-1} a^2 \right).$$

Corollary 3. *On Substituting $p = 1$, $q = 1$ and $V_0 = F_0 = 0$, $V_1 = F_1 = 1$, the above results reduce to*

$$\det K = (-1)^n.$$

Theorem 12.4. (Lucas Formula, 1876). If $a = 0$, $b = 1$, $p = q = 1$, then $V(n)$ reduces to Lucas formula, 1876 (see page 175].

Theorem 6. *If $n \geq 2$ is any integer and $\{V_n\}$ defined in (2), then*

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r V_{r+m}$$
$$= \frac{1}{(\alpha - \beta)} \left[(1-\alpha)^n \beta^m (V_1 - \beta V_0) - (1-\beta)^n \alpha^m (V_1 - \alpha V_0) \right]$$

Proof. We have

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r V_{r+m}$$
$$= \frac{1}{(\alpha - \beta)} (1-\alpha)^n (V_1 - \beta V_0) \beta^m - \frac{1}{(\alpha - \beta)} (1-\beta)^n (V_1 - \alpha V_0) \alpha^m$$
$$= \frac{1}{(\alpha - \beta)} \left[(1-\alpha)^n \alpha^m (V_1 - \beta V_0) - (1-\beta)^n \beta^m (V_1 - \alpha V_0) \right],$$

where α and β are the roots of the equation $x^2 - px - q = 0$. $\square$

Corollary 4. *On substituting $p = 1$, $q = 1$ and $a = F_0 = 0$, $b = F_1 = 1$, the above results are reduced to*

$$\det K = (-1)^n,$$

$$\sum_{i=0}^{n} \binom{n}{r} (-1)^r F_{r+m} = (-1)^{m+1} \left(\frac{\alpha^{n-m} - \beta^{n-m}}{\alpha - \beta} \right) = (-1)^{m+1} F_{n-m},$$

where α and β are the roots of the equation $x^2 - x - 1 = 0$.

Theorem 7. *If $n \geq 2$ is any integer and $\{V_n\}$ defined in (2), then*

$$\sum_{n=2}^{\infty} \left(\frac{1}{V_{n-1}} - \frac{1}{V_{n+1}} \right) = \frac{1}{V_1} + \frac{1}{V_2} = \frac{1}{b} + \frac{1}{pb + aq}$$

Proof. We have

$$\begin{aligned}
\sum_{n=2}^{\infty} \left(\frac{1}{V_{n-1}} - \frac{1}{V_{n+1}} \right) &= \left(\frac{1}{V_1} - \frac{1}{V_3} \right) + \left(\frac{1}{V_2} - \frac{1}{V_4} \right) + \left(\frac{1}{V_3} - \frac{1}{V_5} \right) + \cdots \\
&= \frac{1}{V_1} + \frac{1}{V_2} \\
&= \frac{1}{b} + \frac{1}{pb + aq}.
\end{aligned}$$

□

Corollary 5. *On substituting $p = 1$, $q = 1$ and $a = F_0 = 0$, $b = F_1 = 1$, the above results are reduced to*

$$\sum_{n=2}^{\infty} \left(\frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \right) = \frac{1}{F_1} + \frac{1}{F_2} = \frac{1}{1} + \frac{1}{1} = 2.$$

Theorem 8. *If $n \geq 1$ is any integer then using the Binet's formula in (4), we have*

$$\begin{aligned}
V_n - \alpha V_{n-1} &= \beta^{n-1} (b - \alpha a), \\
V_n - \beta V_{n-1} &= \alpha^{n-1} (b - \beta a),
\end{aligned}$$

where α and β are the roots of the equation $x^2 - px - q = 0$.

Proof. Using the Binet's formula (4), for $n \geq 1$ we have

$$\begin{aligned}
V_n - \alpha V_{n-1} = a \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) + (b - (\alpha + \beta) a) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \\
- \alpha \left(a \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) + (b - (\alpha + \beta) a) \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) \right).
\end{aligned}$$

On simplification we obtain

$$V_n - \alpha V_{n-1} = \beta^{n-1} (b - \alpha a).$$

Similarly

$$V_n - \beta V_{n-1} = \alpha^{n-1} (b - \beta a),$$

where α and β are the roots of the equation $x^2 - px - q = 0$. □

Corollary 6. *On substituting $p = 1$, $q = 1$ and $a = F_0 = 0$, $b = F_1 = 1$, in the above expression we obtain the corresponding expression for n-th classical Fibonacci number [5].*

$$F_n - \alpha F_{n-1} = \beta^{n-1},$$
$$F_n - \beta F_{n-1} = \alpha^{n-1},$$

where α and β are the roots of the equation $x^2 - x - 1 = 0$.

Theorem 9 (Linearization)**.** *If $n \geq 1$ is any integer, then using the Binet's formula in (4) we have*

$$\beta^n = \frac{1}{V_1}\left(\beta V_n + qV_{n-1} - qa\right),$$
$$\alpha^n = \frac{1}{V_1}\left(\alpha V_n + qV_{n-1} - qa\right),$$

where α and β are the roots of the equation $x^2 - px - q = 0$.

Proof. Using the expressions obtained in the above theorem,

$$V_n - \alpha V_{n-1} = \beta^{n-1}\left(V_1 - \alpha a\right),$$
$$V_n - \beta V_{n-1} = \alpha^{n-1}\left(V_1 - \beta a\right).$$

We have

$$\beta^n = \frac{1}{V_1}\left(\beta V_n + qV_{n-1} - qa\right),$$
$$\alpha^n = \frac{1}{V_1}\left(\alpha V_n + qV_{n-1} - qa\right),$$

where α and β are the roots of the equation $x^2 - px - q = 0$. □

Corollary 7. *On substituting $p = 1$, $q = 1$ and $a = F_0 = 0$, $b = F_1 = 1$, in the above expression we obtain the corresponding expression for n-th classical Fibonacci number [5]*

$$F_n - \alpha F_{n-1} = \beta^{n-1},$$
$$F_n - \beta F_{n-1} = \alpha^{n-1},$$

where α and β are the roots of the equation $x^2 - x - 1 = 0$.

3. Discussion and conclusion

In mathematics, the most fascinating sequence of number is the Classical Fibonacci sequence. For mathematicians, it offers various opportunities for new inferences. In this paper, an advanced generalization of the Fibonacci sequence is introduced, where unlike other generalizations, its parameters a, b, p, and q for initial conditions and recurrence relations can be any real numbers. Considering this generalization of the Fibonacci sequence, we present the generalized sequence's term formula, generating function, Binet's formula and some well-known identities in their generalized form. Any reader can employ this generalization for any existing or new identities.

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